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Vortices in Oscillating Liquid

Lord Rayleigh

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which can be produced in an ice-crystal at right angles to the optic axis very closely obeys the laws of viscous flow. If this view should be sustained, it is not only true, as Professor H. A. Miers shows, that a liquid of low viscosity may have a crystalline structure, but also that, as in the case of crystalline ice, a solid may be liquid along one plane only.

From the equation to the curve on Diagram I, the viscosity of an ice-crystal in a direction at right angles to the optic axis is about 2×10^{10} at the freezing point. At this temperature the viscosity of a glacier is about $12{,}500 \times 10^{10}$, or 6250 times as great. The effective viscosity of a glacier is therefore due in a great measure to some other consideration. The optic axes of glacier grains are at all angles and they lock each other.

I am much indebted to Mr. P. H. Parr for the assistance he has rendered in calculating the results from the figures given.

Vortices in Oscillating Liquid.

By Lord Rayleigh, O.M., Pres. R.S.

(Received July 25, 1908.)

In a paper "on the Circulation of Air observed in Kundt's Tubes, and on some Allied Acoustical Problems,"[*] I applied the equations of viscous incompressible fluid to show that the effect of the bottom of the containing vessel was to generate permanent vortices in the vibrating fluid. It was remarkable that the intensity of the vortical motion, when fully established, proved to be independent of the magnitude of the viscosity, so that the effects could not be eliminated by merely supposing the viscosity to become extremely small. The expression found for the vortices was simple. The horizontal component u of the primary motion near the bottom being $u = u_0 \cos kx \cos nt$, the component velocities of the vortical motion are

$$u' = \tfrac{3}{8}\frac{u_0^2 \sin 2kx}{\mathrm{V}} e^{-2ky}(1-2ky),$$

$$-v' = \tfrac{3}{8}\frac{u_0^2 \cos 2kx}{\mathrm{V}} e^{-2ky}\, 2ky,$$

y being measured upwards from the bottom, and $\mathrm{V}\,(= n/k)$ the velocity of propagation of waves of the length in question. According to these

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[*] 'Phil. Trans.,' vol. 175, p. 1, 1883; 'Scientific Papers,' vol. 2, p. 239.

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expressions, the vertical motion is downwards over the places where u has its greatest alternating values. In the case of water contained in a tank and vibrating in its simplest mode, the theoretical motion is downwards in the middle and upwards at the ends. To guard against misinterpretation, it may be well to add that quite close to the bottom the motion, as calculated, is of a quite different character.

In a recent paper,* Mrs. Ayrton has examined, with much experimental skill, the vortices arising when water oscillates in a narrow tank, and has obtained results which differ somewhat widely from what are indicated in the above formulæ. Near the bottom, and especially when the depth is small, there are indeed vortices of this character; but, in general, the most conspicuous feature consists of vortices revolving in the opposite direction, the water rising in the middle of the tank and falling at the ends. The first thought that occurred to me was that Mrs. Ayrton's vortices might be due to defect of freedom in the surface, such as might be supposed to arise from a greasy film opposing extensions and contractions; but in some experiments that I tried, the vortical motion did not seem to be much influenced by cleansing the surface, and the question was suggested as to whether the free surface itself might not originate vortices in somewhat the same way as the bottom does, and more potently on account of the greater velocities of the primary motion there prevailing. I do not remember whether I had any clear view on this question when I wrote the former paper. Vortices originating otherwise than at the bottom were ignored, but I may not have intended to exclude their possibility.

The present paper consists mainly of an attempt to answer the question thus suggested. The investigation is limited to the case of deep water, and even then is rather complicated. If the calculations are correct, we are to conclude that a free surface does *not* generate vortices of this kind, at least if we suppose the viscosity small and include only the square of the motion. How then are the conspicuous vortices, observed by Mrs. Ayrton, to be explained? One might attribute them to the ends of the tank, acting much as the bottom does in my former investigation. In the latter case the seat of the forces is near the parts lying midway between the middle of the tank and the ends ($kx = \pm\frac{1}{4}\pi$), and their effect is to push the neighbouring fluid in the direction away from the place of greatest motion ($kx = 0$). A like action at the ends of the tank would push the neighbouring fluid downwards, and thus generate vortices revolving in the observed direction. An objection to this view lies in an observation by Mrs. Ayrton on water oscillating in a long tank with several subdivisions (p. 255). Since no solid walls are situated at the intermediate nodes, it may be thought that the action at the ends would be insufficient to establish the whole system of vortices. Probably this would be so. But another influence acting in the same direction may arise from the *faces* of the somewhat narrow tanks employed. It would seem that at the nodes, $kx = \pm\frac{1}{2}(2m+1)\pi$, the friction in moving up and down along the faces might have the same general effect as the up-and-down motion along the solid walls which constitute the ends.

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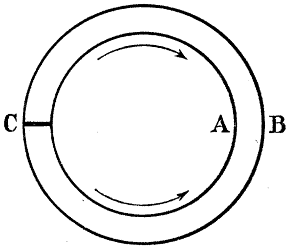
* "On the Non-Periodic or Residual Motion of Water moving in Stationary Waves," 'Roy. Soc. Proc.,' A, vol. 80, p. 252, 1908.

But I must confess that some observations made with the help of Mrs. Sidgwick were not favourable to this view. The ends of the tank were eliminated by using the annular space included between two coaxal cylinders, A, B (beakers), but the vortical motion did not seem to be diminished. The insertion of a strip of glass C held vertically across the annulus, and thus virtually restoring one end, did not make much, if any, difference. This experiment seems to exclude the explanation depending upon the action of the *ends*, and that which would attribute the effect to the *faces* is difficult to reconcile with the highly localised character of the effect, which at first seems to be limited to the immediate neighbourhood of the ends. Can it be that the true explanation would require the retention of terms of higher order than the *square* of the motion?



I may mention that on more than one occasion I have witnessed the reversed movement described by Mrs. Ayrton on p. 259, "as if a set of water springs had been wound up, and now proceeded to unwind themselves." I presume that the spring depends upon gravity, acting upon unequal densities in the fluid, due either to temperature or to variations in the amount of powder held in suspension. Another possible explanation would lie in the effect of surface contamination.

In the usual notation the equations of motion in two dimensions are

$$\frac{1}{\rho}\frac{dp}{dx} = -\frac{du}{dt} + \nu\nabla^2 u - \frac{1}{2}\frac{d(u^2+v^2)}{dx} - v\left(\frac{du}{dy} - \frac{dv}{dx}\right), \qquad (1)$$

$$\frac{1}{\rho}\frac{dp}{dy} = -\frac{dv}{dt} + \nu\nabla^2 v - \frac{1}{2}\frac{d(u^2+v^2)}{dy} - u\left(\frac{du}{dy} - \frac{dv}{dx}\right) - g, \qquad (2)$$

where y is measured vertically upwards, and $\nu (= \mu/\rho)$ is the kinematic viscosity. Since the fluid is supposed to be incompressible,

$$\frac{du}{dx}+\frac{dv}{dy} = 0; \tag{3}$$

or what is equivalent,

$$u = d\psi/dy, \qquad v = -d\psi/dx, \tag{4}$$

ψ being the stream-function.

In virtue of (4), we have in (1), (2)

$$\frac{du}{dy}-\frac{dv}{dx} = \nabla^2\psi. \tag{5}$$

It may be well to commence with the comparatively simple question of stationary waves, carried to the second order of approximation, when viscosity is neglected. If we eliminate p from (1) and (2), putting at the same time $\nu = 0$, we find

$$-\frac{d}{dt}\nabla^2\psi = u\frac{d\nabla^2\psi}{dx}+v\frac{d\nabla^2\psi}{dy}. \tag{6}$$

As was to be expected from the general theory of a frictionless fluid, the solution of (6) to any order of approximation is

$$\nabla^2\psi = 0. \tag{7}$$

We now assume that the motion is periodic with respect to x—proportional, say, to $\cos kx$, so far as the first approximation is concerned. Thus for this approximation we take

$$\psi = \mathrm{A}e^{ky}\cos kx \cos nt, \tag{8}$$

the term in e^{-ky}, otherwise admissible, being excluded by the consideration that all motion must vanish when $y = -\infty$, inasmuch as the fluid is supposed infinitely deep. Corresponding to (8),

$$u = d\psi/dy = k\mathrm{A}e^{ky}\cos kx\cos nt \tag{9}$$

$$v = -d\psi/dx = k\mathrm{A}e^{ky}\sin kx\cos nt, \tag{10}$$

so that

$$u^2+v^2 = k^2\mathrm{A}^2e^{2ky}\cos^2 nt. \tag{11}$$

In virtue of (5), (7), the equations of pressure become

$$\frac{1}{\rho}\frac{dp}{dx} = -\frac{du}{dt}-\tfrac{1}{2}\frac{d(u^2+v^2)}{dx},$$

$$\frac{1}{\rho}\frac{dp}{dy} = -\frac{dv}{dt}-\tfrac{1}{2}\frac{d(u^2+v^2)}{dy}-g;$$

and thus

$$p/\rho = -gy+n\mathrm{A}e^{ky}\sin kx\sin nt-\tfrac{1}{2}k^2\mathrm{A}^2e^{2ky}\cos^2 nt+f(t), \tag{12}$$

where $f(t)$ denotes a function of t which is arbitrary so far as the differential equations of pressure are concerned. The pressure at the surface is to be found from (12) by putting $y = \eta$, where η is the elevation of the surface at the point in question. The relation between η and u, v is thus required accurately to the second order of small quantities.

The differential relation* is

$$\frac{d\eta}{dt} = v - u\frac{d\eta}{dx}. \tag{13}$$

To the first order we have $d\eta/dt$ equal simply to the value of v at the surface, so that by (10)

$$\eta = kn^{-1}\text{A}\sin kx \sin nt, \tag{14}$$

the origin of y being in the undisturbed surface. This value of η may be used in the small terms of (13), and thus to a second approximation

$$\eta = \frac{k\text{A}}{n}\sin kx \sin nt + \frac{k^3\text{A}^2}{4n^2}\cos 2kx \cos 2nt + \text{F}(x), \tag{15}$$

where $\text{F}(x)$ is an arbitrary function of x of the second order of small quantities.

We are now prepared to substitute for y its value η in (12). In the principal term we must use the complete value of η from (15). In the second term, already containing A as a factor, the first approximation for η suffices, while in the third term we may put $\eta = 0$. The third term thus becomes a function of t only, and may be regarded as cancelled by $f(t)$. We find

$$\frac{p}{\rho} = \text{A}\left(n - \frac{gk}{n}\right)\sin kx \sin nt - \frac{k^2\text{A}^2}{4}\cos 2kx$$
$$+ \frac{k^2\text{A}^2}{4}\left(1 - \frac{gk}{n^2}\right)\cos 2kx \cos 2nt - g\,\text{F}(x). \tag{16}$$

In free vibrations the condition to be satisfied at the surface is $p = 0$. The annulment of the term in nt requires that

$$n^2 = gk, \tag{17}$$

and the same well-known relation suffices to annul the term in $2nt$. The surface condition is satisfied without any addition to ψ, if besides satisfying (17) we identify $g\,\text{F}(x)$ with $-\frac{1}{4}k^2\text{A}^2\cos 2kx$. Thus, writing A′ for $k\text{A}/n$, we obtain, as the complete value of η,

$$\eta = \text{A}'\sin kx \sin nt + \tfrac{1}{4}k\text{A}'^2\cos 2kx \cos 2nt - \tfrac{1}{4}k\text{A}'^2\cos 2kx. \tag{18}$$

We now proceed with the consideration of the problem when viscosity

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[*] *Cf.* Lamb's 'Hydrodynamics,' § 10.

is retained. Eliminating p from (1) and (2), we get, with use of (3) and (5),

$$\nabla^4\psi - \frac{1}{\nu}\frac{d}{dt}\nabla^2\psi = \frac{u}{\nu}\frac{d\nabla^2\psi}{dx} + \frac{v}{\nu}\frac{d\nabla^2\psi}{dy}. \tag{19}$$

The terms on the right hand of (19) are of the second order in the amplitude of vibration, and thus for the first approximation we have simply

$$\nabla^4\psi - \frac{1}{\nu}\frac{d}{dt}\nabla^2\psi = 0. \tag{20}$$

The solution of (20) may be written

$$\psi = \psi_1 + \psi_2, \tag{21}$$

where
$$\nabla^2\psi_1 = 0, \qquad \left(\nabla^2 - \frac{1}{\nu}\frac{d}{dt}\right)\psi_2 = 0. \tag{22}$$

We now introduce the suppositions that in the first approximation ψ_1, ψ_2 are proportional to $\cos kx$, and also to e^{int}. The wave-length along x is $2\pi/k$, and the period τ is $2\pi/n$. The equations (20) now become

$$\left(\frac{d^2}{dy^2} - k^2\right)\psi_1 = 0, \qquad \left(\frac{d^2}{dy^2} - k^2 - \frac{in}{\nu}\right)\psi_2 = 0, \tag{23}$$

by which ψ_1 and ψ_2 are to be determined as functions of y. If we write

$$k'^2 = k^2 + in/\nu, \tag{24}$$

we have, as the most general solutions of (23),

$$\psi_1 = \mathrm{A}'e^{ky} + \mathrm{B}'e^{-ky}, \qquad \psi_2 = \mathrm{C}'e^{k'y} + \mathrm{D}'e^{-k'y}; \tag{25}$$

but if the real part of k' is taken positive, the terms in B′ and D′ are excluded when the fluid is treated as infinitely deep. Thus for our purpose

$$\psi_1 = \mathrm{A}\,e^{ky}\cos kx\,e^{int}, \qquad \psi_2 = \mathrm{C}\,e^{k'y}\cos kx\,e^{int}, \tag{26}$$

where A and C are now absolute constants, real or complex. We shall presently find it convenient to suppose C real. From (4) we now find

$$u = (k\mathrm{A}e^{ky} + k'\mathrm{C}e^{k'y})\cos kx\,e^{int}, \tag{27}$$

$$v = (k\mathrm{A}e^{ky} + k\mathrm{C}e^{k'y})\sin kx\,e^{int}; \tag{28}$$

and since to this approximation $d\eta/dt = v$,

$$\eta = (k/in)(\mathrm{A}+\mathrm{C})\sin kx\,e^{int}. \tag{29}$$

If we omit the terms of the second order in (1), we find

$$\frac{1}{\rho}\frac{dp}{dx} = -\frac{d}{dy}\left(\frac{d\psi_1}{dt} + \frac{d\psi_2}{dt}\right) + \nu\frac{d}{dy}(\nabla^2\psi_1 + \nabla^2\psi_2) = -\frac{d}{dy}\frac{d\psi_1}{dt},$$

in virtue of (22); and in like manner

$$\frac{1}{\rho}\frac{dp}{dy} = \frac{d}{dx}\frac{d\psi_1}{dt} - g.$$

Hence
$$\frac{p}{\rho} = -gy + \int\left(\frac{d^2\psi_1}{dx\,dt}\,dy - \frac{d^2\psi_1}{dy\,dt}\,dx\right), \tag{30}$$
the expression to be integrated being a perfect differential by (22).

Applying (30) to the present case, we find
$$p/\rho = -gy - in\mathrm{A}e^{ky}\sin kx\,e^{int}. \tag{31}$$

At the surface we are to suppose $y = \eta$, and η may be neglected in the term already multiplied by A. Thus at the surface
$$\frac{p}{\rho} = i\left\{\frac{gk}{n}(\mathrm{A}+\mathrm{C}) - n\mathrm{A}\right\}\sin kx\,e^{int}. \tag{32}$$

Now that we have to reckon with viscosity, the stress conditions at the surface can no longer be expressed merely by p. In the usual notation, applicable also in the theory of elastic solids, p_{xx}, p_{yy} denote normal tractions across faces perpendicular to x and y respectively, while p_{xy} denotes the tangential traction which acts parallel to y across the face perpendicular to x, or the equal traction parallel to x across the face perpendicular to y. The expressions for these tractions are*
$$p_{xx} = -p + 2\mu\frac{du}{dx}, \qquad p_{yy} = -p + 2\mu\frac{dv}{dy}, \tag{33}$$
$$p_{xy} = \mu\left(\frac{dv}{dx} + \frac{du}{dy}\right). \tag{34}$$

When viscosity was neglected, we were able to suppose that the surface of the fluid was entirely free from imposed force. Under such circumstances the vibrations of a viscous fluid could not be maintained. If n is to be real, some maintaining forces are necessary. We will suppose that these forces are exclusively normal in their character, and accordingly make $p_{xy} = 0$; for it is to be observed that in the present approximation a direction parallel to the surface may be identified with the horizontal. By (27), (28), (34) we find, making $y = 0$,
$$2k^2\mathrm{A} + (k^2 + k'^2)\,\mathrm{C} = 0, \tag{35}$$
as the condition of no tangential force at the surface. Or, if we substitute for k'^2 its value from (24), (35) becomes
$$\mathrm{A} + \left(1 + \frac{in}{2k^2\nu}\right)\mathrm{C} = 0. \tag{36}$$

For the normal traction at the surface we have from (33),
$$\begin{aligned} n\,p_{yy}/\rho = {} & \mathrm{A}\,\{i\,(n^2 - kg) + 2\nu k^2 n\}\sin kx\,e^{int} \\ & + \mathrm{C}\,\{-ikg + 2\nu kk'n\}\sin kx\,e^{int}, \end{aligned} \tag{37}$$

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* *Cf.* Lamb's 'Hydrodynamics,' § 314.

and by (36) the expression for η in (29) may be written

$$\eta = -\frac{C}{2k\nu} \sin kx\, e^{int}. \tag{38}$$

These equations constitute the complete symbolical solution of the problem of infinitely small stationary waves maintained by purely normal surface pressures in a fluid of any degree of viscosity.*

In preparing to pass to real quantities, it is simplest to suppose $C = -2k\nu$, so that

$$\eta = \sin kx\, e^{int}. \tag{39}$$

A is then given explicitly by (36), and on substitution in (37) we get

$$p_{yy}/\rho = \{g - n^2/k + 4i\nu nk + 4\nu^2 k^2 (k - k')\} \sin kx\, e^{int}. \tag{40}$$

The passage to real quantities is now only complicated by the term in k'. If the viscosity be small, this term may be omitted, and we may take

$$p_{yy}/\rho = \{g - n^2/k + 4i\nu nk\} \sin kx\, e^{int} = (g - n^2/k)\,\eta + 4\nu k\,.\,d\eta/dt, \tag{41}$$

giving the normal traction necessary to maintain the waves represented by $\eta = \sin kx \cos nt$. If $n^2 = gk$, the part of p_{yy} in the same phase as η disappears, and

$$p_{yy} = 4\mu k\,.\,d\eta/dt \tag{42}$$

simply. It is to be remembered that p_{yy} is a *traction*. If, as is usual in hydrodynamics, we use pressure (p'), we see that the pressure has its maximum value when the surface is *falling* fastest, as was to be expected.

The accurate expression for the real part of (40) may, of course, be formed. If in (24) we put

$$k^2 = P^2 \cos 2\alpha, \qquad n/\nu = P^2 \sin 2\alpha, \tag{43}$$

then

$$k' = P \cos \alpha + i\, P \sin \alpha. \tag{44}$$

It is unnecessary to write down the actual form of the real part of (40). In most applications an approximate value of k' suffices. On account of the smallness of ν, n/ν is very large in comparison with k^2, that is to say, the thickness of the stratum through which the tangential motion can be propagated in time τ is very small relatively to the wave-length λ. We may, therefore, usually neglect k^4 in the equation

$$P^4 = k^4 + n^2/\nu^2, \tag{45}$$

and take simply

$$P^2 = n/\nu. \tag{46}$$

Again,

$$(\sin \alpha - \cos \alpha)^2 = 1 - \sin 2\alpha = \tfrac{1}{2} k^4 \nu^2 / n^2, \tag{47}$$

so that the difference between cos α and sin α may often be neglected.

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* *Cf.* Basset's 'Hydrodynamics,' § 520; Lamb, *loc. cit.*, § 332.

It appears that the terms neglected in (40) when (41) is substituted are of the order $\nu^{3/2}$.

In proceeding to a second approximation we have first to calculate the terms of the second order forming the right-hand member of (19), using the values found in the first approximation. In these

$$\nabla^2\psi = \nabla^2\psi_2 = \frac{1}{\nu}\frac{d\psi_2}{dt} = \frac{in\mathrm{C}}{\nu}e^{k'y}\cos kx\, e^{int},$$

whence, the imaginary part being rejected,

$$\frac{d\nabla^2\psi}{dx} = \frac{kn\mathrm{C}}{\nu}\sin kx\, e^{\mathrm{P}\cos\alpha\,.\,y}\sin(nt+\mathrm{P}\sin\alpha\,.\,y) \tag{48}$$

$$\frac{d\nabla^2\psi}{dy} = -\frac{n\mathrm{PC}}{\nu}\cos kx\, e^{\mathrm{P}\cos\alpha\,.\,y}\sin(nt+\mathrm{P}\sin\alpha\,.\,y+\alpha). \tag{49}$$

Also, in real quantities by (27), (28), (36),

$$u = \mathrm{C}\cos kx\left\{k\,e^{ky}\left(-\cos nt+\frac{n}{2k^2\nu}\sin nt\right)\right.$$
$$\left.+\mathrm{P}\,e^{\mathrm{P}\cos\alpha\,.\,y}\cos(nt+\mathrm{P}\sin\alpha\,.\,y+\alpha)\right\}, \tag{50}$$

$$v = k\mathrm{C}\sin kx\left\{e^{ky}\left(-\cos nt+\frac{n}{2k^2\nu}\sin nt\right)\right.$$
$$\left.+e^{\mathrm{P}\cos\alpha\,.\,y}\cos(nt+\mathrm{P}\sin\alpha\,.\,y)\right\}. \tag{51}$$

Hence
$$\frac{2\nu\left(u\,d\nabla^2\psi/dx+v\,d\nabla^2\psi/dy\right)}{nk\mathrm{C}^2\sin kx\cos kx\, e^{\mathrm{P}\cos\alpha\,.\,y}}$$
$$= -k\,e^{ky}\left\{\sin(2nt+\mathrm{P}\sin\alpha\,.\,y)+\sin(\mathrm{P}\sin\alpha\,.\,y)\right\}$$
$$+\frac{n\,e^{ky}}{2k\nu}\left\{\cos(\mathrm{P}\sin\alpha\,.\,y)-\cos(2nt+\mathrm{P}\sin\alpha\,.\,y)\right\}$$
$$+\mathrm{P}\,e^{ky}\left\{\sin(2nt+\mathrm{P}\sin\alpha\,.\,y+\alpha)+\sin(\mathrm{P}\sin\alpha\,.\,y+\alpha)\right\}$$
$$-\frac{n\mathrm{P}\,e^{ky}}{2k^2\nu}\left\{\cos(\mathrm{P}\sin\alpha\,.\,y+\alpha)-\cos(2nt+\mathrm{P}\sin\alpha\,.\,y+\alpha)\right\}$$
$$-2\mathrm{P}\,e^{\mathrm{P}\cos\alpha\,.\,y}\sin\alpha. \tag{52}$$

By (19) the equation with which we have to deal is

$$\nabla^4\psi-\frac{1}{\nu}\frac{d}{dt}\nabla^2\psi = \frac{nk\mathrm{C}^2\sin 2kx}{4\nu^2}\times e^{\mathrm{P}\cos\alpha\,.\,y}\times\text{right-hand member of (52).} \tag{53}$$

It will be observed that in (52) or (53) the terms on the right, regarded as functions of t, are either independent of t or circular functions of $2nt$. Corresponding terms, proportional also to $\sin 2kx$, will appear in the direct integral of (53), and in addition we must include a "complementary function" representing, so far as required, the complete integral of (53) when the second member is made equal to zero. This part contains the terms of the first approximation (26), which now represent themselves; and we must also be prepared to admit terms of the second order proportional to sin $2kx$, and either independent of time or involving $2nt$. In the former case the differential equation reduces to

$$\nabla^4\psi = \left(\frac{d^2}{dy^2}-4k^2\right)^2\psi = 0, \tag{54}$$

giving as the solution applicable to deep water,

$$\psi = \sin 2kx\,(\mathrm{H}+\mathrm{K}y)\,e^{2ky}, \tag{55}$$

where H and K are constants. A term in K would represent vortices of the kind found in the former paper to arise from the action of the bottom (when the liquid is not too deep); and one of the principal objects of the present investigation is to ascertain whether these terms occur as the result of the conditions operative at the free surface. It may be recalled that though such vortices could not arise in an ideal frictionless fluid, their magnitude, when fully established, may be independent of the amount of the friction. In view of the complication of the problem it must suffice to limit the investigation to the case of *small* viscosity, the question being whether vortices can be maintained when the viscosity is reduced without limit.

On the right of (52) there are nine terms, of which five are independent of t. But they are not of equal importance. Since P is of the order $\nu^{-\frac{1}{2}}$, the leading term is the seventh in order. So far as this term is concerned,

$$\left(\frac{d^2}{dy^2}-4k^2\right)^2\psi = -\frac{n^2\mathrm{P}\mathrm{C}^2\sin 2kx}{8k\nu^3}\,.\,e^{ky}\,e^{\mathrm{P}\cos\alpha\,.\,y}\cos(\mathrm{P}\sin\alpha\,.\,y+\alpha).$$

It is now convenient to revert to complex quantities, regarding $e^{\mathrm{P}\cos\alpha\,.\,y}\cos(\mathrm{P}\sin\alpha\,.\,y+\alpha)$ as the real part of $e^{k'y+i\alpha}$. Thus

$$\left(\frac{d^2}{dy^2}-4k^2\right)^2\psi = -\frac{k'n^2\,\mathrm{C}^2\sin 2kx}{8k\nu^3}\,.\,e^{(k'+k)\,y},$$

giving
$$\psi = -\frac{k'n^2\,\mathrm{C}^2\sin 2kx}{8k\nu^3\,\{(k'+k)^2-4k^2\}^2}\,e^{(k'+k)\,y}. \tag{56}$$

To form the surface condition, representing the evanescence of tangential force, we shall require the expression of

$$\frac{du}{dy}+\frac{dv}{dx} = \frac{d^2\psi}{dy^2}-\frac{d^2\psi}{dx^2} = \{(k'+k)^2+4k^2\}\,\psi,$$

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in which, since C^2 is already involved as a factor, we may put $y = 0$. Thus

$$\frac{du}{dy}+\frac{dv}{dx} = -\frac{k'n^2C^2\sin 2kx\,\{(k'+k)^2+4k^2\}}{8k\nu^3\,\{(k'+k)^2-4k^2\}^2}, \tag{57}$$

from which the imaginary part is to be rejected.

In tracing the value of (57) as ν diminishes, we get ultimately

$$-\frac{n^2C^2\sin 2kx}{8kk'\nu^3}. \tag{58}$$

In this C^2/ν^2 is of the order A^2, and k' is of order $\nu^{-\frac{1}{2}}$, so that (58) is of order $\nu^{-\frac{1}{2}}A^2$, becoming infinite in comparison with A^2, as ν diminishes without limit. We shall find, however, that this infinite term is compensated by another, to be brought forward later. For our purpose we must retain all terms which do not vanish with ν in comparison with A^2. Hence, with sufficient approximation,

$$\frac{k'\,\{(k'+k)^2+4k^2\}}{\nu\,\{(k'+k)^2-4k^2\}^2} = \frac{k'}{\nu\,(k'+k)^2} = \frac{1}{\nu k'}-\frac{2k}{\nu k'^2} = \frac{1}{\nu k'}-\frac{2k}{in+\nu k^2} = \frac{1}{\nu k'}+\frac{2ik}{n},$$

of which the second term, being purely imaginary, is to be rejected. Thus

$$\frac{du}{dy}+\frac{dv}{dx} = \text{real part of } -\frac{n^2C^2\sin 2kx}{8kk'\nu^3} = -\frac{n^2C^2\cos\alpha\sin 2kx}{8k\nu^3 P}. \tag{59}$$

Referring back to (52) and having regard to the orders of the various terms in respect of ν, we see that the only other term independent of t which needs be retained is the third in order of arrangement. From this term we obtain in like manner,

$$\psi = \frac{n^2C^2\sin 2kx}{8\nu^3}\,\frac{e^{(k'+k)y}}{\{(k'+k)^2-4k^2\}^2}, \tag{60}$$

and when $y = 0$,

$$\frac{du}{dy}+\frac{dv}{dx} = \frac{n^2C^2\sin 2kx}{8\nu^3}\,\frac{(k'+k)^2+4k^2}{\{(k'+k)^2-4k^2\}^2} = \frac{n^2C^2\sin 2kx}{8\nu^3k'^2} \tag{61}$$

with sufficient approximation.

In (61), $\nu k'^2$, so far as it need be retained, is a pure imaginary, and accordingly there is no contribution from this source. We are left therefore with (59) as the complete contribution of the direct integral of (53) when $\nu = 0$, so far as the terms independent of t are concerned.

In a similar manner we may treat the terms in $2nt$. We find for the leading term

$$\nabla^4\psi-\frac{1}{\nu}\frac{d}{dt}\nabla^2\psi = \text{real part of } \frac{k'\,.\,n^2C^2\sin 2kx}{8k\nu^3}\,e^{(k'+k)y}\,e^{2int};$$

so that, if we finally reject the imaginary part,

$$\frac{du}{dy}+\frac{dv}{dx} = \frac{n^2C^2\sin 2kx\,e^{2int}}{8k\nu^2}\left(-\frac{1}{\nu k'}+\frac{2ik}{n}\right), \tag{62}$$

when $y = 0$. And for the only other term which it is necessary to retain,

$$\frac{du}{dy}+\frac{dv}{dx}=\frac{n\mathrm{C}^2}{8\nu^2}\sin 2kx \sin 2nt. \tag{63}$$

We have now sufficiently complete expressions for $du/dy+dv/dx$ so far as it results from the direct integral of (53). But to these we have to add terms arising from the complementary function. In this there must, at any rate, be included the terms in nt found in the first approximation. From (27), (28) we get, when y is small,

$$\frac{du}{dy}+\frac{dv}{dx}=\cos kx \,.\, e^{int}\,\{2k^2\mathrm{A}+(k^2+k'^2)\,\mathrm{C}+2k^2\mathrm{A}\,.\,ky+(k^2+k'^2)\,\mathrm{C}\,.\,k'y\}, \tag{64}$$

and in this we are to substitute for y the value η appropriate to the surface. In the first approximation the terms containing y were neglected, and the surface condition gave

$$2n^2\mathrm{A}+(k^2+k'^2)\,\mathrm{C}=0. \tag{65}$$

This relation must still hold, approximately at any rate. Using it in the small terms of (64), we get

$$(k'^2+k^2)\,\mathrm{C}\,(k'-k)\,y\cos kx\, e^{int},$$

in which we may neglect k in comparison with k'. Passing to real quantities, we find

$$-\frac{n\mathrm{PC}}{\nu}\,y\cos kx\sin(nt+\alpha).$$

The real value of y, or η from (38), is

$$\eta=-\frac{\mathrm{C}}{2k\nu}\sin kx\cos nt, \tag{66}$$

so that this part of (64) becomes

$$\frac{du}{dy}+\frac{dv}{dx}=\frac{n\mathrm{PC}^2}{8k\nu^2}\sin 2kx\,\{\sin(2nt+\alpha)+\sin\alpha\}. \tag{67}$$

Thus for the terms independent of t we get, from (59), (67),

$$\frac{du}{dy}+\frac{dv}{dx}=\frac{n\mathrm{C}^2\sin 2kx}{8k\nu^2}\left\{-\frac{n\cos\alpha}{\mathrm{P}\nu}+\mathrm{P}\sin\alpha\right\}. \tag{68}$$

The factor within braces vanishes with ν, as may be proved from (43), the two infinite terms cancelling without finite residue. Thus altogether $du/dy+dv/dx$ vanishes with ν so far as the terms independent of t are concerned.

The terms in $2nt$ are of less interest. We find, in the same way, when ν is diminished without limit for the complete value of $du/dy+dv/dx$ at the surface,

$$-\frac{n\,\mathrm{C}^2\sin 2kx\sin 2nt}{8\nu^2}. \tag{69}$$

But we are not yet in a position to apply the condition which must be satisfied at the surface, viz., that the tangential force shall there vanish; for we must remember that the surface can no longer be treated as parallel to $y = 0$. If θ be the angle which the surface at the point under consideration makes with $y = 0$, the formulæ of transformation are

$$p_{x'x'} = \cos^2\theta\, p_{xx} + \sin^2\theta\, p_{yy} + \sin 2\theta\, p_{xy},$$
$$p_{y'y'} = \sin^2\theta\, p_{xx} + \cos^2\theta\, p_{yy} - \sin 2\theta\, p_{xy},$$
$$p_{x'y'} = (\cos^2\theta - \sin^2\theta)\, p_{xy} + \sin\theta\cos\theta\,(p_{yy} - p_{xx}).$$

For the present purpose θ may be regarded as a small quantity, equal to $\pm d\eta/dx$, whose square may be neglected, and we may take

$$p_{x'y'} = p_{xy} + \theta\,(p_{yy} - p_{xx}). \qquad (70)$$

It is $p_{x'y'}$, and not p_{xy}, which is to be made to vanish to the second order of the vibration.

The expressions for p_{xx}, etc., have already been given in (33), (34). Substituting the values of u, v of the first approximation, we find, when ν is small,

$$-2\left(\frac{du}{dx} - \frac{dv}{dy}\right) = \frac{2n\mathrm{C}}{\nu}\sin kx \sin nt, \qquad (71)$$

and

$$\frac{d\eta}{dx} = -\frac{\mathrm{C}}{2\nu}\cos kx \cos nt. \qquad (72)$$

It appears, then, that so far as the terms independent of t are concerned, there is no difference between $p_{x'y'}$ and p_{xy}, and since we have already seen that p_{xy} vanishes, it follows that the surface condition of no tangential force is satisfied to a second approximation, without the addition of any further terms (55), such as would represent permanent vortices. Accordingly, no such vortices exist.

As regards terms in $2nt$, we find in addition to (69) another term of the same form derived from (71) and (72). In a solution complete to the second order these terms would need to be compensated by the introduction of new second-order terms in ψ of the form

$$\psi = (\mathrm{L}e^{2ky} + \mathrm{M}e^{k''y})\sin 2kx\, e^{2int}, \qquad (73)$$

where

$$k''^2 = 4k^2 + 2in/\nu; \qquad (74)$$

but it is scarcely necessary for our purpose to define them further. Neither does it seem worth while to express at length the equation of pressure when the second-order terms are included. The particular case of no viscosity already considered illustrates the procedure. Terms in the expression of the pressure which are independent of t are balanced by corresponding terms in η not affecting the velocities.
